

On the p -regularized trust region subproblem

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Abstract The p -regularized subproblem (p-RS) is a regularisation technique in computing a Newton-like step for unconstrained optimization, which globally minimizes a local quadratic approximation of the objective function while incorporating with a weighted regularisation term $\frac{\sigma}{p}\|x\|^p$. The global solution of the p -regularized subproblem for $p = 3$, also known as the cubic regularization, has been characterized in literature. In this paper, we resolve both the global and the local non-global minimizers of (p-RS) for $p > 2$ with necessary and sufficient optimality conditions. Moreover, we prove a parallel result of Martínez [13] that the (p-RS) for $p > 2$, analogous to the trust region subproblem, can have at most one local non-global minimizer. When the (p-RS) is subject to a fixed number m additional linear inequality constraints, we show that the uniqueness of the local solution of the (p-RS) (if exists at all), especially for $p = 4$, can be applied to solve such an extension in polynomial time.

Keywords Newton method · Regularization · Trust-region subproblem · Local minimizer · Extended Trust-region subproblem

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1 Introduction

For an unconstrained optimization problem to minimize f over \mathbb{R}^n , Newton's method has an attractive local convergence property near a second order critical point. Ensuring the global convergence for Newton's method with an analyzable computational complexity, however, requires modifications to guarantee a *sufficient* descent at each step. Unlike the Levenberg-Marquardt type of methods or most quasi-Newton methods which always maintain a positive-definite approximate Hessian of f , the p -regularized subproblem minimizes globally the second order Taylor's polynomial of f plus a weighted (by σ) higher order regularization term. The subproblem takes the following model

$$(\text{p-RS}) \quad \min_{x \in \mathbb{R}^n} \left\{ g(x) = \frac{1}{2} x^T H x + c^T x + \frac{\sigma}{p} \|x\|^p \right\}, \quad (1)$$

where $\sigma > 0$, $p > 2$, and H is the Hessian of f at any iterate, regardless of its definiteness. It is often assumed that f is smooth enough to have a symmetric Hessian and to obtain the desire global convergence. At each iterate, if the global minimizer of (p-RS) renders a satisfactory decrease in the value of f , it is accepted; but rejected otherwise with an increase in σ to enhance the regularization force.

In literature, (p-RS) with $p = 3$ is known as the cubic regularization which is the most common choice among all others. The idea of the cubic regularization was first due to Griewank [8] and later was considered by many authors with thorough global convergence and complexity analysis. See Nesterov and Polyak [15]; Weiser Deuffhard and Erdmann [17]; and Cartis, Gould and Toint [2]. When $p = 4$, (p-RS) reduces to a form of the double well potential function which has many applications in solid mechanics and quantum mechanics [5, 18]. Gould, Robinson and Thorne [7] studied (p-RS) for a general $p > 2$ in comparison with the the trust-region subproblem

$$(\text{TRS}) \quad \min \frac{1}{2} x^T H x + c^T x \quad (2)$$

$$\text{s.t. } \|x\|^2 \leq \Delta, \quad x \in \mathbb{R}^n. \quad (3)$$

Our paper characterizes (p-RS) completely for any $p > 2$ by extending (i) the necessary and sufficient global optimality conditions for $p = 3$ in [2]; (ii) the analysis using the secular function (to be specified later) for $p = 4$ in [18]; and (iii) a necessary global optimality condition for $p > 2$ in [7]. Some generalization is, nevertheless, non-trivial in mathematical skills. We summarize the main results as follows.

- Theorem 1 of the paper (*cf.* Theorem 3.1 in [2] for $p = 3$; Theorem 2 in [7] for the necessary part of $p > 2$): The point x^* is a global minimizer of (p-RS) for $p > 2$ if and only if

$$(H + \sigma \|x^*\|^{p-2} I) x^* = -c; \quad H + \sigma \|x^*\|^{p-2} I \succeq 0.$$

- Theorem 2 (*cf.* the trust region subproblem in [11]): Let k be the multiplicity of the smallest eigenvalue α_1 of H , i.e.,

$$\alpha_1 = \dots = \alpha_k < \alpha_{k+1} \leq \dots \leq \alpha_n.$$

Then, the set of the global minimizers of (p-RS) is either a singleton or a k -dimensional sphere centered at $(0, \dots, 0, -\frac{c_{k+1}}{\alpha_{k+1}-\alpha_1}, \dots, -\frac{c_n}{\sigma_n-\sigma_1})$ with

the radius $\sqrt{\left(\frac{\alpha_1}{\sigma}\right)^{\frac{2}{p-2}} - \sum_{i=k+1}^n \frac{c_i^2}{(\alpha_i-\alpha_1)^2}}$.

- Theorem 3 (*cf.* Theorem 2 in [18] for $p = 4$): The point \underline{x} is a local-nonglobal minimizer of (p-RS) for $p > 2$ if and only if

$$\underline{x} = -(H + \sigma \underline{t}^* I)^{-1} c, \quad (4)$$

where \underline{t}^* is a root of the secular function

$$h(t) = \|(H + \sigma t I)^{-1} c\|^2 - t^{\frac{2}{p-2}}, \quad t \in \left(\max\left\{-\frac{\alpha_2}{\sigma}, 0\right\}, -\frac{\alpha_1}{\sigma}\right) \quad (5)$$

such that $h'(\underline{t}^*) > 0$.

- Theorem 4 (*cf.* the trust region subproblem in [13]; the double well potential function in [18]): The subproblem (p-RS) with $p > 2$ has at most one local non-global minimizer.

Notice that, the secular function for (TRS) (*cf.* $h(t)$ in (5)) is defined by

$$\phi(\lambda) = \|(H + \lambda I)^{-1} c\|^2.$$

Martínez [13] proved that, if \underline{x} is a local-nonglobal minimizer of (TRS), then \underline{x} satisfies $(H + \lambda^* I)\underline{x} = -c$ with $\lambda^* \in (-\alpha_2, -\alpha_1)$, $\lambda^* \geq 0$ and $\phi'(\lambda^*) \geq 0$. However, to the best of our knowledge, the necessary condition $\phi'(\lambda^*) \geq 0$ is not known to be sufficient for (TRS) or not.

Finally, as an application, we study (p-RS) subject to m linear inequality constraints of the following form:

$$(\text{p-RS}_m) \quad \min \frac{1}{2} x^T H x + c^T x + \frac{\sigma}{p} \|x\|^p \quad (6)$$

$$\text{s.t. } l_i \leq a_i^T x \leq u_i, \quad i = 1, \dots, m, \quad (7)$$

where $l_i \leq u_i \in \mathbb{R}$ for $i = 1, \dots, m$. We first show that the NP-hard k -dispersion-sum problem

$$(\text{KDSP}) \quad d^* = \min x^T (-D) x \quad (8)$$

$$\text{s.t. } e^T x = k, \quad x \in \{0, 1\}^n \quad (9)$$

can be reduced to a special case of (p-RS $_{n+1}$) with $p = 4$. It indicates that solving the class of subproblems $\bigcup_{m>n} (\text{p-RS}_m)$ with $p = 4$ is also NP-hard. However, for any fixed m , by Theorem 4 that there is at most one local non-global minimizer for (p-RS $_m$) with $p = 4$, we show that it can be solved in polynomial time. Notice that there is an analogy called the extended trust

region subproblem which adds linear inequality constraints to (TRS). Polynomial solvability has been recently proved by Bienstock and Michalka [1], and independently by Hsia and Sheu [11].

Notations. Let $v(\cdot)$ denote the optimal value of problem (\cdot) . For any symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P \succ (\succeq) 0$ means that P is positive (semi)definite. The determinant of P is denoted by $\det(P)$ whereas the identity matrix of order n by I . For a vector $x \in \mathbb{R}^n$, $\text{Diag}(x)$ is a diagonal matrix with diagonal components being x_1, \dots, x_n . For a number $\beta \in \mathbb{R}$, $\text{sign}(\beta) = \frac{\beta}{|\beta|}$ if $\beta \neq 0$, otherwise $\text{sign}(\beta) = 0$.

2 Characterization of the Global Minimizers

We first observe that the objective function $g(x)$ of (p-RS) is coersive, i.e.,

$$\lim_{\|x\| \rightarrow +\infty} g(x) = +\infty.$$

Consequently, the global minimizer of (p-RS) always exists. The starting point of the analysis is the first order and the second order necessary conditions for any local minimizer of g .

Lemma 1 *Assume that \underline{x} is a local minimizer of (p-RS), $p > 2$. It holds that*

$$\nabla g(\underline{x}) = (H + \sigma \|\underline{x}\|^{p-2} I) \underline{x} + c = 0, \quad (10)$$

$$\nabla^2 g(\underline{x}) = (H + \sigma \|\underline{x}\|^{p-2} I) + \sigma(p-2) \|\underline{x}\|^{p-4} \underline{x} \underline{x}^T \succeq 0, \quad (11)$$

where ∇g , $\nabla^2 g$ denote the gradient and the Hessian of $g(x)$, respectively.

The next theorem shows that, a local minimizer \underline{x} becomes global if and only if $H + \sigma \|\underline{x}\|^{p-2} I \succeq 0$. The necessity has been shown by Theorem 2 in [7]. We only proves the sufficiency here.

Theorem 1 *The point x^* is a global minimizer of (p-RS) for $p > 2$ if and only if it is a critical point satisfying $\nabla g(x^*) = 0$ and $H + \sigma \|x^*\|^{p-2} I \succeq 0$. Moreover, the ℓ_2 norms of all the global minimizers are equal.*

Proof. If $x^* = 0_n$, then $\sigma \|x^*\|^{p-2} = 0$ so that $c = -(H + \sigma \|x^*\|^{p-2} I)x^* = 0$ and $H = H + \sigma \|x^*\|^{p-2} I \succeq 0$. Consequently, $x^T H x \geq 0$, $\forall x \in \mathbb{R}^n$. It follows that $x^* = 0_n$ is a global minimizer since

$$g(x) = \frac{1}{2} x^T H x + c^T x + \frac{\sigma}{p} \|x\|^p \geq \frac{\sigma}{p} \|x\|^p > 0 = g(0), \quad \forall x \neq 0_n = x^*.$$

Now we assume $x^* \neq 0_n$, i.e., $\|x^*\| > 0$. Define $Q = H + \sigma \|x^*\|^{p-2} I$. According to the assumption, $Q \succeq 0$. Then, for any $x \in \mathbb{R}^n$ and $x \neq x^*$, it holds that

$$\begin{aligned} g(x) &= \frac{1}{2} x^T H x + c^T x + \frac{\sigma}{p} \|x\|^p \\ &= \frac{1}{2} x^T Q x + c^T x - \frac{1}{2} (\sigma \|x^*\|^{p-2}) x^T x + \frac{\sigma}{p} \|x\|^p \\ &= \frac{1}{2} x^T Q x + c^T x + \frac{\sigma}{p} \|x^*\|^p \left(\left(\frac{\|x\|^2}{\|x^*\|^2} \right)^{\frac{p}{2}} - \frac{p}{2} \frac{\|x\|^2}{\|x^*\|^2} \right) \end{aligned} \quad (12)$$

Define $f(t) = t^{\frac{p}{2}}$, $p > 2$. It is strictly convex for $t > 0$. Therefore,

$$f(t) = t^{\frac{p}{2}} \geq f(1) + f'(1)(t - 1) = 1 + \frac{p}{2}(t - 1), \quad \forall t > 0.$$

By substituting t with $\frac{\|x\|^2}{\|x^*\|^2}$, we have

$$\left(\frac{\|x\|^2}{\|x^*\|^2} \right)^{\frac{p}{2}} - \frac{p}{2} \frac{\|x\|^2}{\|x^*\|^2} \geq 1 - \frac{p}{2}.$$

Then,

$$g(x) \geq \frac{1}{2}x^T Qx + c^T x + \frac{\sigma}{p}\|x^*\|^p \left(1 - \frac{p}{2}\right). \quad (13)$$

By $Q \succeq 0$, the lower bounding function of g in the right hand side of (13) is convex quadratic in terms of x . Since x^* satisfies $(H + \sigma\|x^*\|^{p-2}I)x^* = Qx^* = -c$, x^* is a global minimizer of the convex function in the right hand side of (13). As a consequence,

$$g(x) \geq \frac{1}{2}(x^*)^T Qx^* + c^T x^* + \frac{\sigma}{p}\|x^*\|^p \left(1 - \frac{p}{2}\right) = g(x^*)$$

and x^* is a global minimizer of (p-RS).

Finally, if $\|\hat{x}\| = \|x^*\|$, from (12) it can be seen that

$$g(\hat{x}) = \frac{1}{2}(\hat{x})^T Q\hat{x} + c^T \hat{x} + \frac{\sigma}{p}\|x^*\|^p \left(1 - \frac{p}{2}\right).$$

Then, \hat{x} is also a global minimizer of (p-RS) if and only if $\nabla g(\hat{x}) = 0$ and $\|\hat{x}\| = \|x^*\|$. □

To study the hidden convexity of (RS), without loss of generality, we assume H is diagonal, i.e.,

$$H = \text{Diag}(\alpha_1, \dots, \alpha_n), \quad \alpha_1 \leq \dots \leq \alpha_n. \quad (14)$$

Otherwise, let $H = U\Sigma U^T$ be the eigenvalue decomposition of H . Let $y = U^T x$. Notice that $\|y\| = \|U^T x\| = \|x\|$. We obtain a diagonal (RS) in terms of y .

Proposition 1 *Suppose H is diagonal. Let x^* be the global minimizer of (RS), then we have*

$$c_i x_i^* \leq 0, \quad i = 1, \dots, n.$$

Proof. Let $\tilde{x} = (-x_1^*, x_2^*, x_3^*, \dots, x_n^*)$. According to the definition of x^* , we have

$$0 \geq g(x^*) - g(\tilde{x}) = c_1(x_1^* - \tilde{x}_1) = 2c_1 x_1^*.$$

A similar argument applies for the other components. □

Now we establish the hidden convexity of (RS). According to Proposition 1, (RS) is equivalent to

$$\begin{aligned} \min \quad & \sum_{i=1}^n \left\{ \frac{\alpha_i}{2} x_i^2 + c_i x_i \right\} + \frac{\sigma}{p} \left(\sum_{i=1}^n x_i^2 \right)^{\frac{p}{2}} \\ \text{s.t.} \quad & c_i x_i \leq 0, \quad i = 1, \dots, n. \end{aligned} \quad (15)$$

Introducing the nonlinear one-to-one map:

$$x_i = \begin{cases} \sqrt{z_i}, & \text{if } c_i \leq 0, \\ -\sqrt{z_i}, & \text{if } c_i > 0, \end{cases} \quad i = 1, \dots, n, \quad (16)$$

(RS) is equivalent to the following convex program:

$$\begin{aligned} \min \quad & - \sum_{i=1}^n |c_i| \sqrt{z_i} + \frac{1}{2} \sum_{i=1}^n \alpha_i z_i + \frac{\sigma}{p} \left(\sum_{i=1}^n z_i \right)^{\frac{p}{2}} \\ \text{s.t.} \quad & z_i \geq 0, \quad i = 1, \dots, n. \end{aligned} \quad (17)$$

Since (17) is strictly convex when $p > 2$, again, we can see that $\sum_{i=1}^n z_i^* = \|x^*\|^2$ is unique where z^*, x^* are any global minimizers of (17) and (RS), respectively.

Before ending this section, we show that the set of the global minimizers of (RS), denoted by $\{x^*\}$, is either a singleton or a k -dimensional sphere where k is the multiplicity of the smallest eigenvalue α_1 , i.e.,

$$\alpha_1 = \dots = \alpha_k < \alpha_{k+1} \leq \dots \leq \alpha_n.$$

According to Theorem 1, we first have

$$\sigma \|x^*\|^{p-2} + \alpha_1 \geq 0.$$

- Suppose $c_1^2 + \dots + c_k^2 > 0$. It follows from (10) that

$$\sigma \|x^*\|^{p-2} + \alpha_1 > 0.$$

Solving (10) yields

$$x_i^* = \frac{-c_i}{\sigma \|x^*\|^{p-2} + \alpha_i}, \quad i = 1, \dots, n.$$

By summing all $(x_i^*)^2$, we can derive that

$$t^* = \|x^*\|^{p-2}$$

is a nonnegative root of the following secular function on a specific open interval:

$$h(t) = \sum_{i=1}^n \frac{c_i^2}{(\sigma t + \alpha_i)^2} - t^{\frac{2}{p-2}}, \quad t \in \left(-\frac{\alpha_1}{\sigma}, +\infty \right). \quad (18)$$

Since $\lim_{t \rightarrow \max\{-\frac{\alpha_1}{\sigma}, 0\}} h(t) > 0$, $\lim_{t \rightarrow +\infty} h(t) = -\infty$ and $h(t)$ is strictly decreasing on $(-\frac{\alpha_1}{\sigma}, +\infty)$, the secular function $h(t)$ has a unique solution t^* on $(\max\{-\frac{\alpha_1}{\sigma}, 0\}, +\infty)$. In this case, x^* defined by

$$x_i^* = \frac{-c_i}{\sigma t^* + \alpha_i}, \quad i = 1, \dots, n \quad (19)$$

is the unique global minimum solution of (RS).

- Suppose $c_1^2 + \dots + c_k^2 = 0$. The secular function (18) reduces to

$$h(t) = \sum_{i=k+1}^n \frac{c_i^2}{(\sigma t + \alpha_i)^2} - t^{\frac{2}{p-2}}, \quad t \in \left(-\frac{\alpha_1}{\sigma}, +\infty\right). \quad (20)$$

There are two cases.

- (1) $\alpha_1 > 0$. Since $h(0) \geq 0$, (20) has a unique nonnegative solution t^* . Then, x^* satisfying (19) is the unique global minimizer.
- (2) $\alpha_1 \leq 0$ and $h(-\frac{\alpha_1}{\sigma}) > 0$. Therefore, (20) has a unique nonnegative solution t^* and hence x^* satisfying (19) is the unique global minimizer.
- (3) $\alpha_1 \leq 0$ and $h(-\frac{\alpha_1}{\sigma}) \leq 0$. In this case, (20) has no solution. By Theorem 1, any x^* satisfying

$$(x_1^*)^2 + \dots + (x_k^*)^2 = -h\left(-\frac{\alpha_1}{\sigma}\right), \quad (21)$$

$$x_i^* = -\frac{c_i}{\alpha_i - \alpha_1}, \quad i = k+1, \dots, n \quad (22)$$

is a global minimizer. Namely, the global minimum solution set forms a k -dimensional sphere centered at $(0, \dots, 0, -\frac{c_{k+1}}{\alpha_{k+1} - \alpha_1}, \dots, -\frac{c_n}{\alpha_n - \alpha_1})$

with the radius $\sqrt{\left(\frac{\alpha_1}{\sigma}\right)^{\frac{2}{p-2}} - \sum_{i=k+1}^n \frac{c_i^2}{(\alpha_i - \alpha_1)^2}}$.

3 Characterization of the Local-Nonglobal Minimizer

In this section, we establish the necessary and sufficient optimality condition for the local-nonglobal minimizer of (RS).

Let $\alpha_1 \leq \dots \leq \alpha_n$ be the eigenvalues of H . Throughout this section, we assume $\alpha_1 < 0$. That is, $H \not\geq 0$. Otherwise, (RS) is a convex minimization problem and hence has no local-nonglobal minimizer.

Lemma 2 *Suppose $H \not\geq 0$. Then 0_n is not a local minimizer of (RS).*

Proof. Suppose 0_n is a local minimizer of (RS). Then the necessary optimality conditions (10)-(11) imply that

$$c = 0, \quad H \succeq 0,$$

which is contradiction.

Lemma 3 Suppose $n \geq 2$. Let \underline{x} be a local minimizer of (RS) . It holds that

$$\sigma\|\underline{x}\|^{p-2} + \alpha_2 \geq 0. \quad (23)$$

Furthermore, if $\alpha_1 < \alpha_2$, then

$$\sigma\|\underline{x}\|^{p-2} + \alpha_2 > 0. \quad (24)$$

Proof. Without loss of generality, we can assume H is a diagonal matrix, i.e., $H = \text{Diag}(\alpha_1, \dots, \alpha_n)$. Suppose the statement (23) is not true, then $\sigma\|\underline{x}\|^{p-2} + \alpha_2 < 0$.

$$\sigma\|\underline{x}\|^{p-2} + \alpha_1 \leq \sigma\|\underline{x}\|^{p-2} + \alpha_2 < 0.$$

Let e_1 and e_2 be the first two columns of I , respectively. We consider the following two cases.

(a) Suppose $\underline{x}_1 = e_1^T \underline{x} = 0$. It follows from the necessary condition (11) that

$$0 \leq e_1^T (\sigma(p-2)\|\underline{x}\|^{p-4} \underline{x} \underline{x}^T + \sigma\|\underline{x}\|^{p-2} I + H) e_1 = \sigma\|\underline{x}\|^{p-2} + \alpha_1 < 0,$$

which is a contradiction.

(b) Suppose $\underline{x}_1 = e_1^T \underline{x} \neq 0$. It follows from the necessary condition (11) that

$$\begin{aligned} 0 &\leq ((-\underline{x}_2)e_1 + (\underline{x}_1)e_2)^T (\sigma(p-2)\|\underline{x}\|^{p-4} \underline{x} \underline{x}^T \\ &\quad + \sigma\|\underline{x}\|^{p-2} I + H) ((-\underline{x}_2)e_1 + (\underline{x}_1)e_2) \\ &= (\sigma\|\underline{x}\|^{p-2} + \alpha_1)(\underline{x}_2)^2 + (\sigma\|\underline{x}\|^{p-2} + \alpha_2)(\underline{x}_1)^2 < 0, \end{aligned}$$

which is a contradiction.

Therefore, the statement (23) holds true.

Now we assume $\alpha_1 < \alpha_2$ and suppose that the statement (24) is not true. Then we have

$$\sigma\|\underline{x}\|^{p-2} + \alpha_2 = 0, \quad (25)$$

with which the necessary optimality condition (11) becomes

$$\begin{aligned} 0 &\preceq \sigma(p-2)\|\underline{x}\|^{p-4} \underline{x} \underline{x}^T + \sigma\|\underline{x}\|^{p-2} I + H \\ &= \sigma(p-2)\|\underline{x}\|^{p-4} \underline{x} \underline{x}^T + H - \alpha_2 I. \end{aligned} \quad (26)$$

Consequently, the first two leading principal minors of the matrix in (26) are nonnegative, i.e.,

$$\sigma(p-2)\|\underline{x}\|^{p-4} \underline{x}_1^2 + \alpha_1 - \alpha_2 \geq 0 \quad (27)$$

and

$$\begin{aligned} &\det \left\{ \sigma(p-2)\|\underline{x}\|^{p-4} \begin{bmatrix} \underline{x}_1^2 & \underline{x}_1 \underline{x}_2 \\ \underline{x}_1 \underline{x}_2 & \underline{x}_2^2 \end{bmatrix} + \begin{bmatrix} \alpha_1 - \alpha_2 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \sigma(p-2)\|\underline{x}\|^{p-4} (\alpha_1 - \alpha_2) \underline{x}_2^2 \geq 0. \end{aligned} \quad (28)$$

Since $\alpha_1 - \alpha_2 < 0$, the inequalities (27) and (28) imply $\underline{x}_1 \neq 0$ and $\underline{x}_2 = 0$, respectively. Then it follows from the necessary optimality condition (10) that obtain that $c_2 = 0$ and

$$\underline{x}_1 = \frac{-c_1}{\sigma \|\underline{x}\|^{p-2} + \alpha_1} = \frac{c_1}{\alpha_2 - \alpha_1}.$$

Without loss of generality, we assume that $c_1 > 0$, which implies that $\underline{x}_1 > 0$. Then, according to (25) and the fact $\underline{x}_2 = 0$, we have

$$\underline{x}_1 = \sqrt{\left(\frac{-\alpha_2}{\sigma}\right)^{\frac{2}{p-2}} - \sum_{i=3}^n \underline{x}_i^2}.$$

Consider the following parametric curve in \mathbb{R}^n :

$$\begin{aligned} \gamma(t) &= \{(k(t), t, \underline{x}_3, \dots, \underline{x}_n) | \\ k(t) &= \sqrt{\left(\frac{-\alpha_2}{\sigma}\right)^{\frac{2}{p-2}} - t^2 - \sum_{i=3}^n \underline{x}_i^2} = \sqrt{\underline{x}_1^2 - t^2}, t \in \mathbb{R}\} \end{aligned} \quad (29)$$

where $\gamma(0) = \gamma(\underline{x}_2) = \underline{x}$, i.e., $\gamma(t)$ passes through \underline{x} at $t = 0$. Evaluating $g(x)$ on $\gamma(t)$, we have

$$\begin{aligned} g(\gamma(t)) &= \frac{\sigma}{p} \left(k(t)^2 + t^2 + \sum_{i=3}^n \underline{x}_i^2 \right)^{\frac{p}{2}} + \frac{\alpha_1}{2} k(t)^2 + \frac{\alpha_2}{2} t^2 + \sum_{i=3}^n \frac{\alpha_i}{2} \underline{x}_i^2 + c_1 k(t) + \sum_{i=3}^n c_i \underline{x}_i \\ &= \frac{\sigma}{p} \left(\underline{x}_1^2 + \sum_{i=3}^n \underline{x}_i^2 \right)^{\frac{p}{2}} + \frac{\alpha_1}{2} \underline{x}_1^2 + \sum_{i=3}^n \frac{\alpha_i}{2} \underline{x}_i^2 + \frac{\alpha_2 - \alpha_1}{2} t^2 + c_1 \sqrt{\underline{x}_1^2 - t^2} + \sum_{i=3}^n c_i \underline{x}_i. \end{aligned}$$

Since \underline{x} is a local minimizer of $g(x)$, $t = 0$ must be a local minimum point of $g(\gamma(t))$. However, this conclusion contradicts to the fact that

$$\frac{d}{dt}g(\gamma(0)) = \frac{d^2}{dt^2}g(\gamma(0)) = \frac{d^3}{dt^3}g(\gamma(0)) = 0, \quad \frac{d^4}{dt^4}g(\gamma(0)) = -\frac{3(\alpha_2 - \alpha_1)}{\underline{x}_1^2} < 0.$$

Consequently, the statement (24) holds true under the additional assumption $\alpha_1 < \alpha_2$. \square

As the main result in this section, we establish the necessary and sufficient condition for local-nonglobal minimizer of (RS).

Theorem 2 \underline{x} is a local-nonglobal minimizer of (RS) if and only if

$$\underline{x} = -(\sigma \underline{t}^* I + H)^{-1} c, \quad (30)$$

where \underline{t}^* is a root of the secular function

$$h(t) = \|(\sigma t I + H)^{-1} c\|^2 - t^{\frac{2}{p-2}}, \quad t \in \left(\max \left\{ -\frac{\alpha_2}{\sigma}, 0 \right\}, -\frac{\alpha_1}{\sigma} \right) \quad (31)$$

such that $h'(\underline{t}^*) > 0$.

Proof. Without loss of generality, we can assume H is a diagonal matrix, i.e., $H = \text{Diag}(\alpha_1, \dots, \alpha_n)$. It is sufficient to consider the nontrivial case $n \geq 2$, since for $n = 1$, we will see that it amounts to setting $\alpha_2 = \infty$ in the following proof.

According to Lemma 3 and Theorem 1, the local-nonglobal minimizer \underline{x} of (RS) exists only if

$$-\alpha_2 < \sigma \|\underline{x}\|^{p-2} < -\alpha_1. \quad (32)$$

It follows that the diagonal matrix $\sigma \|\underline{x}\|^{p-2} I + H$ is nonsingular with its first diagonal element being negative and others positive. Solving (10), we obtain

$$\underline{x}_i = \frac{-c_i}{\sigma \|\underline{x}\|^{p-2} + \alpha_i}, \quad i = 1, \dots, n. \quad (33)$$

The necessary optimality condition (11) implies that

$$\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}_1^2 + \sigma \|\underline{x}\|^{p-2} + \alpha_1 \geq 0.$$

Then it follows from the right hand side of (32) that

$$\|\underline{x}\| > 0, \quad (34)$$

and moreover,

$$\underline{x}_1 \neq 0, \quad (35)$$

Putting all \underline{x}_i in (33) together yields

$$\sum_{i=1}^n \frac{c_i^2}{(\sigma \|\underline{x}\|^{p-2} + \alpha_i)^2} = \|\underline{x}\|^2. \quad (36)$$

As a summary of (32), (34) and (36),

$$t^* = \|\underline{x}\|^{p-2}$$

is a root of the following secular function on a specific open interval:

$$h(t) = \sum_{i=1}^n \frac{c_i^2}{(\sigma t + \alpha_i)^2} - t^{\frac{2}{p-2}}, \quad t \in \left(\max \left\{ -\frac{\alpha_2}{\sigma}, 0 \right\}, -\frac{\alpha_1}{\sigma} \right), \quad (37)$$

which is the diagonal version of (37). Notice that each root of $h(t) = 0$ can only correspond to one local-nonglobal minimizer of (RS) due to (33). Taking a simple calculation of (37), we have

$$h'(t) = - \sum_{i=1}^n \frac{2\sigma c_i^2}{(\sigma t + \alpha_i)^3} - \frac{2}{p-2} t^{\frac{4-p}{p-2}}. \quad (38)$$

We notice that the necessary optimality condition (11) is equivalent to

$$\sigma(p-2)\|\underline{x}\|^{p-4}(\Gamma \underline{x})(\Gamma \underline{x})^T + \text{Diag}(-1, 1, \dots, 1) \succeq 0, \quad (39)$$

where

$$\Gamma = \text{Diag} \left(\frac{1}{\sqrt{-\sigma \|\underline{x}\|^{p-2} - \alpha_1}}, \frac{1}{\sqrt{\sigma \|\underline{x}\|^{p-2} + \alpha_2}}, \dots, \frac{1}{\sqrt{\sigma \|\underline{x}\|^{p-2} + \alpha_n}} \right). \quad (40)$$

Since the determinant of the positive semidefinite matrix in (39) is nonnegative, we have

$$\begin{aligned} 0 &\leq \det(\sigma(p-2)\|\underline{x}\|^{p-4}(\Gamma\underline{x})(\Gamma\underline{x})^T + \text{Diag}(-1, 1, \dots, 1)) \\ &= \det(\text{Diag}(-1, 1, \dots, 1)) \times \\ &\quad \det(\sigma(p-2)\|\underline{x}\|^{p-4}\text{Diag}(-1, 1, \dots, 1)(\Gamma\underline{x})(\Gamma\underline{x})^T + I) \\ &= -1 \times (\sigma(p-2)\|\underline{x}\|^{p-4}(\Gamma\underline{x})^T \text{Diag}(-1, 1, \dots, 1)(\Gamma\underline{x}) + 1) \\ &= -\sum_{i=1}^n \frac{\sigma(p-2)\|\underline{x}\|^{p-4}c_i^2}{(\sigma\|\underline{x}\|^{p-2} + \alpha_i)^3} - 1 \\ &= \left(\frac{p}{2} - 1\right)\|\underline{x}\|^{p-4}h'(\|\underline{x}\|^{p-2}) \\ &= \left(\frac{p}{2} - 1\right)\|\underline{x}\|^{p-4}h'(t^*). \end{aligned}$$

It follows from $p > 2$ and (34) that $h'(t^*) \geq 0$. Now, it remains to show that $h'(t^*) > 0$. Suppose this is not true, we have $h'(t^*) = 0$. Therefore, we obtain

$$\begin{aligned} &\det(\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}\underline{x}^T + \sigma\|\underline{x}\|^{p-2}I + H) \\ &= \frac{\det(\sigma(p-2)\|\underline{x}\|^{p-4}(\Gamma\underline{x})(\Gamma\underline{x})^T + \text{Diag}(-1, 1, \dots, 1))}{\det^2(\Gamma)} \\ &= \frac{(\frac{p}{2} - 1)\|\underline{x}\|^{p-4}h'(t^*)}{\det^2(\Gamma)} \\ &= 0 \end{aligned} \quad (41)$$

and thus there is a $u = (u_1, \dots, u_n)^T \neq 0$ such that

$$\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}\underline{x}^T u + (\sigma\|\underline{x}\|^{p-2}I + H)u = 0, \quad (42)$$

or equivalently,

$$u_i = \frac{-\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}_i(u^T \underline{x})}{\sigma\|\underline{x}\|^{p-2} + \alpha_i}, \quad i = 1, 2, \dots, n.$$

Since $u \neq 0$, it holds that

$$u^T \underline{x} \neq 0. \quad (43)$$

Define

$$q(\beta) := g(\underline{x} + \beta u).$$

We can verify that

$$\begin{aligned} q'(\beta) &= \nabla g(\underline{x} + \beta u)u, \\ q''(\beta) &= u^T \nabla^2 g(\underline{x} + \beta u)u, \\ q'''(\beta) &= 3\sigma(p-2)\|\underline{x} + \beta u\|^{p-4}(u^T \underline{x} + \beta u^T u)u^T u \\ &\quad + \sigma(p-2)(p-4)\|\underline{x} + \beta u\|^{p-6}(u^T \underline{x} + \beta u^T u)^3. \end{aligned}$$

The necessary optimality condition (10) implies that $q'(0) = 0$. According to the definition of u , we have $q''(0) = 0$. However, (43) implies that

$$\begin{aligned} (q'''(0))^2 &= \sigma^2(p-2)^2\|\underline{x}\|^{2(p-6)}(u^T \underline{x})^6 \left(3 \frac{(\underline{x}^T \underline{x})(u^T u)}{(u^T \underline{x})^2} + (p-4) \right)^2 \\ &\geq \sigma^2(p-2)^2\|\underline{x}\|^{2(p-6)}(u^T \underline{x})^6(p-1)^2 \\ &> 0, \end{aligned}$$

where the first inequality follows from Cauchy-Schwartz inequality. It contradicts to the fact that \underline{x} is a local minimizer of (RS). Therefore, $h'(\underline{t}^*) > 0$ and the necessary proof is complete.

It remains for us to give the sufficient proof. Let $t^* \in (\max\{-\frac{\alpha_2}{\sigma}, 0\}, -\frac{\alpha_1}{\sigma})$ be a root of the secular function (37) such that $h'(t^*) > 0$. Define \underline{x} as in (30). Then we have

$$\|\underline{x}\|^2 = \sum_{i=1}^n \frac{c_i^2}{(\sigma \underline{t}^* + \alpha_i)^2} = (\underline{t}^*)^{\frac{2}{p-2}},$$

that is, $\underline{t}^* = \|\underline{x}\|^{p-2}$. Consequently, \underline{x} satisfies the first-order necessary optimality condition (10). Moreover, the diagonal matrix $\sigma\|\underline{x}\|^{p-2}I + H$ is nonsingular with positive diagonal elements except for the first one. By Weyl's inequality (see [12], Theorem 4.3.1), we have

$$\begin{aligned} \lambda_i(\nabla^2 g(\underline{x})) &= \lambda_i(\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}\underline{x}^T + \sigma\|\underline{x}\|^{p-2}I + H) \\ &\geq \lambda_1(\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}\underline{x}^T) + \lambda_i(\sigma\|\underline{x}\|^{p-2}I + H) \\ &= \lambda_i(\sigma\|\underline{x}\|^{p-2}I + H) \\ &> 0, \quad \text{for } i = 2, 3, \dots, n, \end{aligned} \tag{44}$$

where $\lambda_i(P)$ is the i th smallest eigenvalue of P . Since $h'(\underline{t}^*) > 0$, by (41), we have

$$\begin{aligned} \prod_{i=1}^n \lambda_i(\nabla^2 g(\underline{x})) &= \det(\nabla^2 g(\underline{x})) \\ &= \det(\sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}\underline{x}^T + \sigma\|\underline{x}\|^{p-2}I + H) \\ &= \frac{(\frac{p}{2} - 1)\|\underline{x}\|^{p-4}h'(\underline{t}^*)}{\det^2(I^*)} \\ &> 0, \end{aligned} \tag{45}$$

Combining (44) with (45), we have

$$\lambda_1(\nabla^2 g(\underline{x})) > 0,$$

or equivalently,

$$\nabla^2 g(\underline{x}) = \sigma(p-2)\|\underline{x}\|^{p-4}\underline{x}\underline{x}^T + \sigma\|\underline{x}\|^{p-2}I + H \succ 0.$$

This is a sufficient condition to guarantee that \underline{x} is a local minimizer of (RS). The proof is complete. \square

Theorem 2 and its proof provide some simple sufficient conditions for having no local-nonglobal minimizer.

Corollary 1 *When one of the following conditions is met:*

- (a) $\alpha_1 \geq 0$;
- (b) $\alpha_1 = \alpha_2$;
- (c) $v^T c = 0$, where v is the eigenvector of H corresponding to α_1 ;

any local minimizer of (RS) is globally optimal.

Proof. In Case (a), $g(x)$ is convex and hence any local minimizer is globally optimal. In Case (b), it is trivial to see that the secular function (37) has no solution. Therefore, according to Theorem 2, the local-nonglobal minimizer does not exist. Suppose (RS) has a local-nonglobal minimizer in Case (c). Let $H = U\Sigma U^T$ be the eigenvalue decomposition of H . Introducing $y = U^T x$, we obtain a diagonal version of (RS) with respect to y :

$$\min_{y \in \mathbb{R}^n} \left\{ g(x) = \frac{1}{2}y^T \Sigma y + \tilde{c}^T y + \frac{\sigma}{p}\|y\|^p \right\}$$

where $\tilde{c} = U^T c$. According to (33) and (35) in the necessary proof of Theorem 2, a necessary condition for the secular function (37) having a solution is that $\tilde{c}_1 \neq 0$. We obtain a contradiction by noting that $\tilde{c}_1 = (U^T c)_1 = v^T c$, where v is the eigenvector of H corresponding to α_1 . \square

The second corollary of Theorem 2 can be regarded as the similar version of Proposition 1 for the local-nonglobal minimizer.

Corollary 2 *Suppose H is diagonal. Let \underline{x} be the local-nonglobal minimizer of (RS), then we have*

$$c_1 \underline{x}_1 > 0, \quad c_i \underline{x}_i \leq 0, \quad i = 2, 3, \dots, n. \quad (46)$$

Proof. Following (30) and (31), we immediately have

$$c_1 \underline{x}_1 \geq 0, \quad c_i \underline{x}_i \leq 0, \quad i = 2, 3, \dots, n.$$

The fact $\underline{x}_1 \neq 0$ is shown in (35) and the statement $c_1 \neq 0$ follows from (35) and (33). \square

As an application of Corollary 2, similar to (17) we see that finding the local-non-global minimizer of (RS) is actually equivalent to globally minimizing the following nonconvex program:

$$\begin{aligned} \min & |c_i| \sqrt{z_i} - \sum_{i=2}^n |c_i| \sqrt{z_i} + \frac{1}{2} \sum_{i=1}^n \alpha_i z_i + \frac{\sigma}{p} \left(\sum_{i=1}^n z_i \right)^{\frac{p}{2}} \\ \text{s.t. } & z_i \geq 0, \quad i = 1, \dots, n. \end{aligned}$$

As the last corollary of Theorem 2, we have

Theorem 3 (RS) with $p > 2$ has at most one local-nonglobal minimizer.

Proof. First we observe that the secular function (31) has the same roots as

$$p(t) = \log \left(\|(\sigma t I + H)^{-1} c\|^2 \right) - \frac{2}{p-2} \log(t), \quad t \in \left(\max \left\{ -\frac{\alpha_2}{\sigma}, 0 \right\}, -\frac{\alpha_1}{\sigma} \right).$$

Without loss of generality, we assume H is diagonal. If $c_1 = 0$, then (RS) has no local-nonglobal minimizer according to Corollary 2. So, we assume $c_1 \neq 0$. Then, we have

$$p''(t) = \frac{\sum_{i=1}^n \frac{6\sigma^2 c_i^2}{(\sigma t + \alpha_i)^4}}{\sum_{i=1}^n \frac{c_i^2}{(\sigma t + \alpha_i)^2}} - \frac{\left(\sum_{i=1}^n \frac{2\sigma c_i^2}{(\sigma t + \alpha_i)^3} \right)^2}{\left(\sum_{i=1}^n \frac{c_i^2}{(\sigma t + \alpha_i)^2} \right)^2}.$$

Define two vectors in \mathbb{R}^n :

$$a = \left(\frac{\sqrt{6}\sigma c_1}{(\sigma t + \alpha_1)^2}, \dots, \frac{\sqrt{6}\sigma c_n}{(\sigma t + \alpha_n)^2} \right)^T, \quad b = \left(\frac{c_1}{\sigma t + \alpha_1}, \dots, \frac{c_n}{\sigma t + \alpha_n} \right)^T.$$

Applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \left(\sum_{i=1}^n \frac{2\sigma c_i^2}{(\sigma t + \alpha_i)^3} \right)^2 &< (a^T b)^2 \\ &\leq (a^T a)(b^T b) \\ &= \left(\sum_{i=1}^n \frac{6\sigma^2 c_i^2}{(\sigma t + \alpha_i)^4} \right) \left(\sum_{i=1}^n \frac{c_i^2}{(\sigma t + \alpha_i)^2} \right). \end{aligned}$$

Therefore, $p''(t) > 0$ for all t such that $p(t)$ is well-defined. It follows that $p(t)$ is strictly convex for $t \in \left(\max \left\{ -\frac{\alpha_2}{\sigma}, 0 \right\}, -\frac{\alpha_1}{\sigma} \right)$. Thus, $p(t)$, as well as $h(t)$, has at most two real roots in this interval. Let $t_1 < t_2$ be the only two roots of $h(t)$. Suppose $h'(t_1) > 0$ and $h'(t_2) > 0$. Then, for sufficiently small $\epsilon \in (0, \frac{t_2 - t_1}{2})$, we have

$$h(t_1 + \epsilon) > h(t_1) = 0, \quad h(t_2 - \epsilon) < h(t_2) = 0.$$

Therefore, there is a $\tilde{t} \in [t_1 + \epsilon, t_2 - \epsilon]$ such that $h(\tilde{t}) = 0$, which is a contradiction. consequently, the secular function $h(t)$ has at most one real root satisfying $h'(t) > 0$. Following Theorem 2, the proof is complete. \square

4 (RS) with linear inequality constraints

In this section, we study (RS_m) (6)-(7). For a special case $p = 4$, we first show (RS_m) is NP-hard when $m > n$. Then, as an application of Theorems 2 and 3, we show (RS_m) can be solved in polynomial time when m is a fixed number.

4.1 NP-hardness

Let $S = \{x \in [0, 1]^n \mid Ax \leq b\}$ be nonempty, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are with integer elements, and for $i = 1, \dots, n$, the i -th row of A , denoted by a_i , satisfies that $\|a_i\|_1 \geq 2$.

Lemma 4 ([10]) *For any vertex $x = (x_1, \dots, x_n)^T$ of the polytope S , if $x \notin \{0, 1\}^n$, then it holds that*

$$x^T(e - x) \geq \begin{cases} \frac{\max_{1 \leq j \leq m} \|a_j\|_\infty - 1}{\max_{1 \leq j \leq m} \|a_j\|_\infty^2}, & \text{if } \max_{1 \leq j \leq m} \|a_j\|_\infty \geq 2, \\ \frac{1}{2}, & \text{if } \max_{1 \leq j \leq m} \|a_j\|_\infty = 1, \end{cases} \quad (47)$$

where e is a vector of dimension n with all components equal to one.

Now, we consider the following k -dispersion-sum problem:

$$(KDSP) \quad d^* = \min x^T(-D)x \quad (48)$$

$$\text{s.t. } e^T x = k, \quad x \in \{0, 1\}^n. \quad (49)$$

It is to locate k facilities at some of n predefined locations by maximizing the distance sum between the k established facilities, where the distance between two facilities i and j is given by a square matrix $D = (d_{ij})$, $i, j = 1, 2, \dots, n$. (KDSP) is NP-hard, even if the distance matrix satisfies the triangle inequality, see [4, 9].

Define the continuous relaxation of (KDSP) as

$$d^c = \min_{e^T x = k, \quad x \in [0, 1]^n} x^T(-D)x.$$

It trivially holds that $d^c \leq d^*$. For any $\theta \geq 4(d^* - d^c)$, we obtain

$$\begin{aligned} & \min_{e^T x = k, \quad x \in [0, 1]^n} \left\{ d(x) := -x^T D x + \theta (e^T x - x^T x)^2 \right\} \\ &= \min \left\{ \min_{e^T x = k, \quad x \in [0, 1]^n \setminus \{0, 1\}^n} d(x), \quad \min_{e^T x = k, \quad x \in \{0, 1\}^n} d(x) \right\} \\ &\geq \min \left\{ \min_{e^T x = k, \quad x \in [0, 1]^n} -x^T D x + \min_{e^T x = k, \quad x \in [0, 1]^n \setminus \{0, 1\}^n} \theta (e^T x - x^T x)^2, \quad d^* \right\} \\ &\geq \min \left\{ d^c + \frac{1}{4}\theta, \quad d^* \right\} \\ &= d^*, \end{aligned}$$

where the second inequality holds since

$$x^T(e - x) \geq \frac{1}{2}, \quad \forall x \in [0, 1]^n \setminus \{0, 1\}^n, \quad e^T x = k,$$

which follows from Lemma 4. Therefore, (KDSP) has been reduced to the following special case of (RS_{n+1}) with $p = 4$:

$$\min -x^T D x + \theta (k - x^T x)^2 = -x^T (D + 2\theta k \cdot I) x + \theta \|x\|^4 + \theta k^2 \quad (50)$$

$$\text{s.t. } e^T x = k, \quad x \in [0, 1]^n. \quad (51)$$

As a summary, we have the following result:

Theorem 4 *When $p = 4$, (RS_m) with $m > n$ is NP-hard.*

4.2 Polynomially Solvable Cases

Consider (RS_m) with $p = 4$ and m being a fixed number. The approach applied in this subsection inherits from [11].

Let X_0^* denote the set of the global minimizers of (RS). According to the discussion at the end of Section 2, X_0^* is either a singleton or a k -dimensional sphere, both can be obtained in polynomial time.

We first check whether $\in X_0^* \cap \{x \mid l_i \leq a_i^T x \leq u_i, i = 1, \dots, m\}$ is empty, which can be done in polynomial time according to the following lemma.

Lemma 5 ([11]) *Let $A \in R^{m \times q}$ and $b \in R^m$, where m is fixed and q is arbitrary. For any given $r > 0$, it is polynomially checkable whether $\{u \in R^q \mid Au \leq b, u^T u = r\}$ is empty. Moreover, if the set is nonempty, a feasible point can be found in polynomial time.*

If $\in X_0^* \cap \{x \mid l_i \leq a_i^T x \leq u_i, i = 1, \dots, m\} \neq \emptyset$, any point in $X_0^* \cap \{x \mid l_i \leq a_i^T x \leq u_i, i = 1, \dots, m\}$, globally solves (RS_m). Otherwise, we find the unique local-nonglobal minimizer of (RS), denoted by \bar{x}_0 , which is obtained in polynomial time according to Theorem 2. Moreover, if

$$l_i < a_i^T \bar{x}_0 < u_i, \quad i = 1, 2, \dots, m, \quad (52)$$

then, \bar{x}_0 is the unique attained solution of the following problem:

$$\begin{aligned} v(\text{RS}_m^0) &:= \min g(x) = \frac{1}{2} x^T H x + c^T x + \frac{\sigma}{4} \|x\|^4 \\ \text{s.t. } &l_i < a_i^T x < u_i, \quad i = 1, 2, \dots, m. \end{aligned} \quad (53)$$

It follows that

$$v(\text{RS}_m) = \min\{v(\text{RS}_m^0), v(\text{RS}_m^{1_1}), v(\text{RS}_m^{1_2}), \dots, v(\text{RS}_m^{m_1}), v(\text{RS}_m^{m_2})\} \quad (54)$$

where $v(\text{RS}_m^{j_1})$ and $v(\text{RS}_m^{j_2})$ ($j = 1, 2, \dots, m$) are defined as follows:

$$\begin{aligned} v(\text{RS}_m^{j_1}) &:= \min g(x) = \frac{1}{2}x^T Hx + c^T x + \frac{\sigma}{4}\|x\|^4 \\ \text{s.t. } &a_j^T x = l_j, \\ &l_i \leq a_i^T x \leq u_i, \quad i = 1, \dots, j-1, j+1, \dots, m, \end{aligned} \quad (55)$$

$$\begin{aligned} v(\text{RS}_m^{j_2}) &:= \min g(x) = \frac{1}{2}x^T Hx + c^T x + \frac{\sigma}{4}\|x\|^4 \\ \text{s.t. } &a_j^T x = u_j, \\ &l_i \leq a_i^T x \leq u_i, \quad i = 1, \dots, j-1, j+1, \dots, m. \end{aligned} \quad (56)$$

Otherwise, (52) does not hold true. Then, we have

$$v(\text{RS}_m) = \min\{v(\text{RS}_m^{1_1}), v(\text{RS}_m^{1_2}), \dots, v(\text{RS}_m^{m_1}), v(\text{RS}_m^{m_2})\}. \quad (57)$$

It remains to show how to solve $(\text{RS}_m^{j_1})$, $j = 1, \dots, m$ as $(\text{RS}_m^{j_2})$ is similarly solved. Our idea is to eliminate one variable using the equation (55) and maintains the same structure as (RS_m) .

Let $P_j \in \mathbb{R}^{n \times (n-1)}$ be a column-orthogonal matrix such that $a_j^T P_j = 0$. Let z_0 be a feasible solution to (55). Then $z_0 - P_j P_j^T z_0$ is also feasible to (55). Using the null-space representation, we have

$$\{x \in \mathbb{R}^n \mid a_j^T x = b_j\} = \{z_0 - P_j P_j^T z_0 + P_j z \mid z \in \mathbb{R}^{n-1}\} \quad (58)$$

and

$$\begin{aligned} \|x\|^4 &= ((z_0 - P_j P_j^T z_0 + P_j z)^T (z_0 - P_j P_j^T z_0 + P_j z))^2 \\ &= (z_0^T (I - P_j P_j^T) (I - P_j P_j^T) z_0 + 2z_0^T (I - P_j P_j^T) P_j z + z^T P_j^T P_j z)^2 \\ &= (z_0^T (I - P_j P_j^T) z_0 + z^T z)^2 \\ &= (z_0^T (I - P_j P_j^T) z_0)^2 + 2(z_0^T (I - P_j P_j^T) z_0) z^T z + \|z\|^4 \end{aligned}$$

We can equivalently express $(\text{RS}_m^{j_1})$ as:

$$\begin{aligned} v(\text{RS}_m^{j_1}) &= \min g(z_0 - P_j P_j^T z_0 + P_j z) \\ \text{s.t. } &l_i \leq a_i^T (z_0 - P_j P_j^T z_0 + P_j z) \leq u_i, \quad i = 1, \dots, m, \quad i \neq j \end{aligned}$$

which is again a special case of (RS_{m-1}) .

Iteratively applying (54) or (57), we will eventually terminate when no linear constraint left. Let s be the smallest number such that any $s+1$ columns of $\{a_1, \dots, a_m\}$ are dependent. By this inductive way, there are at most $m \times (m-1) \times \dots \times (m-s+1)$ regularised subproblems to be solved. Since m is assumed to be fixed, the total number of reduction iterations is bounded by a constant factor of m . We thus have proved that

Theorem 5 *For each fixed m , (RS_m) is polynomially solvable.*

5 Conclusions

In this paper we have characterized the local and global minimizers of the regularised subproblem (RS) in optimization. We first show the existing necessary optimality condition for (RS) in literature is also *sufficient* and the ℓ_2 norm of the global minimizer is *always* unique. The hidden convexity of (RS) is also obtained. Then we establish a necessary and sufficient condition for the local-nonglobal minimizer of (RS). We notice that this condition remains open for the trust-region subproblem. As a corollary, we show (RS) with $p > 2$ has at most *one* local-nonglobal minimizer. As a further application, we show (RS) with $p = 4$ and a fixed number of linear inequality constraints can be solved in polynomial time, while general linear constrained (RS) is shown to be NP-hard. It is unknown what happens when $p \neq 4$.

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